Lesson 1. Sample Paths

1 Course overview

- In this course, we will learn how to model and analyze systems that evolve dynamically over time and whose behavior is stochastic, or uncertain
- The models we will use to analyze such systems are called **stochastic processes**
- Numerous applications:
  - business and economics – consumer behavior, portfolio management, inventory
  - military – enemy movement, personnel dynamics, maintenance and readiness
  - medicine – disease behavior, policy evaluation
- Techniques used: probability, statistics, matrix theory
- This lesson: an example illustrating some basic ideas

2 Sample paths

- A **sample path** is a record of the time-dependent behavior of a system
  - For example, the time that each customer arrives and departs at the drive-thru at Starbucks
- A sample path can be decomposed into **inputs** and **logic**
  - Inputs = arrival times and service times of customers
  - Logic = how the drive-thru operates: number of windows, first-come-first-served, etc.
- A **simulation** generates new sample paths resulting from changes in the input or logic without building a new system
- **Sample path analysis** uses sample paths generated by a simulation to determine system performance measures

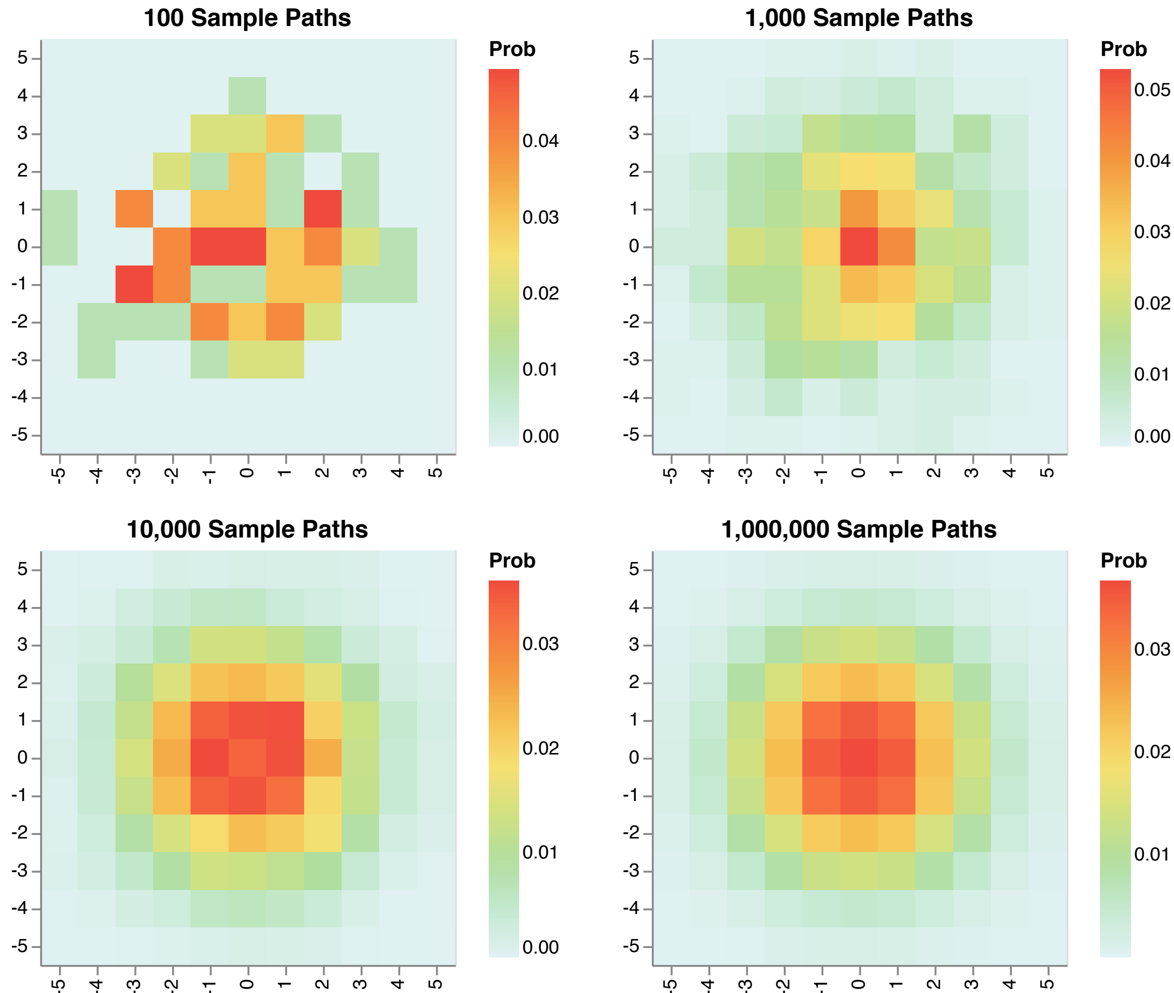
3 Sample path analysis to study submarine behavior

Ballistic missile submarines act as a nuclear deterrence to enemy countries. These submarines move randomly throughout the ocean within a fixed grid assigned by a higher authority. Approximately every 20 minutes, the submarine will turn to change its course in order to clear its *baffles*, the area directly behind the submarine where sonar cannot detect sound.

- Let's consider the movement of a single submarine
- Suppose the submarine operates in an 11×11 grid (see page 5)
- The submarine starts at box $(0, 0)$
- Every 20 minutes, the submarine randomly selects 1 of the 8 cardinal directions (N, NE, E, SE, S, SW, W, NW) and moves one box in that direction
  - For example, after the first 20 minutes, if the submarine selects N, then it will move to box $(0, 1)$

- We want to model the movement of the submarine over time
- Let's model time using the index set $N = \{0, 1, 2, \ldots,\}$ so that each time index represents
- Let S_n denote the position of the submarine at time n
  - For example, $S_0 =$
- Using an 8-sided die, generate a sample path representing the movement of the submarine for the first 100 minutes (see page 5)
  - Repeat this 10 times and record your submarine's 10 final locations
- Did you generate the same sample path multiple times?
- Assume that you have a fair die. What is the probability that the submarine moves N on the first roll?
- Based on the sample paths you generated, what is the probability that the submarine moves N on the first roll? (In other words, what percentage of time did the submarine move N on the first roll in your simulations?)
- How many possible sample paths can be generated?
- In general, what kind of information can you learn from generating many sample paths?

- Suppose we ran this same experiment, but with many more repetitions – for example, generating
    - 100 sample paths
    - 1,000 sample paths
    - 10,000 sample paths
    - 1,000,000 sample paths
- We would get something that looks like this:



- What do you observe in the probability distributions above?

4 What's next?

- In this lesson's example, we generated sample paths to determine an **empirical probability distribution** for the state of the system – i.e., the location of the submarine after 100 minutes
  - An empirical probability distribution is based on the frequency of observed data
- If we make certain reasonable assumptions, we can mathematically analyze sample paths without actually generating them and determine a **theoretical probability distribution** for the state of the system
  - A theoretical probability distribution is based on logic or mathematical formulas
- These kinds of models are the focus of this course – for example:
  - Poisson processes
  - Markov chains
  - Markov processes and queueing systems
- Such models are used in real-world applications
  - For example, the Undersea Warfighting Development Center in Groton, CT uses Markov chains to model the movement of submarines

Roll the 8-sided die to generate 10 sample paths representing the submarine's movements for the first 100 minutes. Use the following key to determine which cardinal direction the submarine should move.

1	2	3	4	5	6	7	8
N	NE	E	SE	S	SW	W	NW

$(-5,5)$	$(-4,5)$	$(-3,5)$	$(-2,5)$	$(-1,5)$	$(0,5)$	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$
$(-5,4)$	$(-4,4)$	$(-3,4)$	$(-2,4)$	$(-1,4)$	$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$
$(-5,3)$	$(-4,3)$	$(-3,3)$	$(-2,3)$	$(-1,3)$	$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$
$(-5,2)$	$(-4,2)$	$(-3,2)$	$(-2,2)$	$(-1,2)$	$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$
$(-5,1)$	$(-4,1)$	$(-3,1)$	$(-2,1)$	$(-1,1)$	$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(-5,0)$	$(-4,0)$	$(-3,0)$	$(-2,0)$	$(-1,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,0)$
$(-5,-1)$	$(-4,-1)$	$(-3,-1)$	$(-2,-1)$	$(-1,-1)$	$(0,-1)$	$(1,-1)$	$(2,-1)$	$(3,-1)$	$(4,-1)$	$(5,-1)$
$(-5,-2)$	$(-4,-2)$	$(-3,-2)$	$(-2,-2)$	$(-1,-2)$	$(0,-2)$	$(1,-2)$	$(2,-2)$	$(3,-2)$	$(4,-2)$	$(5,-2)$
$(-5,-3)$	$(-4,-3)$	$(-3,-3)$	$(-2,-3)$	$(-1,-3)$	$(0,-3)$	$(1,-3)$	$(2,-3)$	$(3,-3)$	$(4,-3)$	$(5,-3)$
$(-5,-4)$	$(-4,-4)$	$(-3,-4)$	$(-2,-4)$	$(-1,-4)$	$(0,-4)$	$(1,-4)$	$(2,-4)$	$(3,-4)$	$(4,-4)$	$(5,-4)$
$(-5,-5)$	$(-4,-5)$	$(-3,-5)$	$(-2,-5)$	$(-1,-5)$	$(0,-5)$	$(1,-5)$	$(2,-5)$	$(3,-5)$	$(4,-5)$	$(5,-5)$